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Coefficient Inequalities for Strongly Close-to-Convex Functions

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Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be a normalized strongly close-to-convex function of order $\alpha > 0$ defined on the unit disk $\mathbb{D}$. This means that there is a normalized convex univalent function φ and $\beta \in \mathbb{R}$ such that

$$\left| \arg \frac{f'(z)}{e^{i\beta}\varphi'(z)} \right| < \frac{\alpha\pi}{2}$$

for $z \in \mathbb{D}$. Then

$$|a_3 - a_2^2| + \tfrac{1}{3}|a_x|^2 \le \tfrac{1}{3}(1 + 4\alpha + 2\alpha^2)$$

and

$$\left|a_3 - \tfrac{2}{3}a_2^2\right| + \tfrac{2}{3}|a_2|^2 \le \frac{1}{3}(3 + 4\alpha + 2\alpha^2)$$

with equality if and only if f is a rotation of

$$F_\alpha(z) = \frac{1}{2(1+\alpha)}\left[\left(\frac{1+z}{1-z}\right)^{1+\alpha} - 1\right].$$

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1. INTRODUCTION

A holomorphic function f defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is called strongly close-to-convex of order $\alpha > 0$ provided f is normalized $(f(0) = 0, f'(0) = 1)$ and there exist $\beta \in \mathbb{R}$ and a normalized convex univalent function φ such that

$$\left| \arg \frac{f'(z)}{e^{i\beta}\varphi'(z)} \right| < \frac{\alpha\pi}{2}$$

for $z \in \mathbb{D}$. Let $\mathrm{SCC}(\alpha)$, $0 \le \alpha \le 1$, denote the class of all strongly close-to-convex functions of order α. The usual class of normalized close-to-convex functions is SCC(1). For $\alpha = 0$ we need to replace strict inequality by "less than or equal to." Then $\mathrm{SCC}(0) = \mathrm{CV}$, the class of normalized convex univalent functions. Then $\mathrm{SCC}(0) \subset \mathrm{SCC}(\alpha)$ for $\alpha > 0$. For $\alpha \in [0, 1]$ all functions in $\mathrm{SCC}(\alpha)$ are univalent.

The function

$$\begin{aligned} F_\alpha(z) &= \frac{1}{2(1+\alpha)} \left[\left(\frac{1+z}{1-z} \right)^{1+\alpha} - 1 \right] \\ &= z + (1+\alpha) z^2 + \frac{1}{3}(3 + 4\alpha + 2\alpha^2) z^3 + \cdots \end{aligned}$$

belongs to $\mathrm{SCC}(\alpha)$ and is known to be extremal for a number of problems. For instance, if $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathrm{SCC}(\alpha)$, then

$$|a_2| \le 1 + \alpha$$

and

$$|a_3| \le \tfrac{1}{3}(3 + 4\alpha + 2\alpha^2).$$

Equality holds in either quality if and only if $f(z) = e^{-i\theta} F_\alpha(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$; that is, f is a rotation of F_α. See [2, Chap. 11] for the interesting history of the coefficient problem for $\mathrm{SCC}(\alpha)$.

The goal of this paper is to establish two sharp coefficient inequalities for the class $\mathrm{SCC}(\alpha)$. The authors were led to such inequalities by their recent work relating to two-point distortion theorems for univalent function [5]. These inequalities yield sharp two-point distortion theorems for nonnormalized strongly close-to-convex functions of order $\alpha \in [0, 1]$. Our main result is the following theorem.

THEOREM 1. *Suppose* $f(z) = z + a_2 z^2 + a_2 z^3 + \cdots \in \mathrm{SCC}(\alpha)$. *Then*

$$\text{(i)} \quad \left|a_3 - a_2^2\right| + \tfrac{1}{3}|a_2|^2 \leq \tfrac{1}{3}(1 + 4\alpha + 2\alpha^2)$$

and

$$\text{(ii)} \quad \left|a_3 - \tfrac{2}{3}a_2^2\right| + \tfrac{2}{3}|a_2|^2 \leq \tfrac{1}{3}(3 + 4\alpha + 2\alpha^2).$$

Equality holds if and only if f is a rotation of F_α.

Inequality (ii) implies the sharp bound on $|a_3|$ over the class $\mathrm{SCC}(\alpha)$. For $\alpha = 0$ the inequality (i) was established for the class CV by Trimble [7]. Also, for $\alpha = 0$ the coefficient bound $|a_2| \leq 1$ for the class CV together with inequality (i) yields inequality (ii). For $\alpha > 0$ the second inequality does not seem to be a consequence of the first.

2. REPRESENTATION FORMULA FOR SCC(α)

A basic tool in the proof of our main theorem is an elementary representation for derivatives of functions in $\mathrm{SCC}(\alpha)$ that was already noted in [1]. If $f \in \mathrm{SCC}(\alpha)$, then $[f'(z)/(e^{i\beta}\varphi'(z))]^{1/\alpha}$ has positive real part in $\mathbb{D}$ if we select the branch at the origin with value $e^{-i\beta/\alpha}$. The fact that this function has positive real part in $\mathbb{D}$ implies that $\cos(\beta/\alpha) > 0$, so we may assume that $\theta = \beta/\alpha \in (-\pi/2, \pi/2)$. Then

$$p(z) = \frac{1}{\cos(\theta)}\left[\left(\frac{f'(z)}{e^{i\beta}\varphi'(z)}\right)^{1/\alpha} + i\sin(\theta)\right]$$

belongs to the class $\mathscr{P}$ of normalized ($p(0) = 1$) holomorphic functions defined on $\mathbb{D}$ with positive real part. Thus, if $f \in \mathrm{SCC}(\alpha)$, then

$$\begin{aligned} f'(z) &= e^{i\alpha\theta}\varphi'(z)[\cos(\theta)p(z) - i\sin(\theta)]^\alpha \\ &= \varphi'(z)\{e^{i\theta}[\cos(\theta)p(z) - i\sin(\theta)]\}^\alpha, \end{aligned}$$

where $\alpha > 0$, $|\theta| < \pi/2$, $\varphi \in \mathrm{CV}$, and $p \in \mathscr{P}$. Conversely, if f' has this form, then $f \in \mathrm{SCC}(\alpha)$. For future reference we note the explicit representation of the function F_α and its rotations. For $e^{-i\theta}F_\alpha(e^{i\theta})$ we have

$$\begin{aligned} F'_\alpha(e^{i\theta}z) &= \varphi'(e^{i\theta}z)p(e^{i\theta}z)^\alpha \\ &= \frac{1}{(1 - e^{i\theta}z)^2}\left(\frac{1 + e^{i\theta}z}{1 - e^{i\theta}z}\right)^\alpha, \end{aligned}$$

where $\varphi(z) = z/(1 - z) \in \mathrm{CV}$ and $p(z) = (1 + z)/(1 - z) \in \mathscr{P}$.

From the representation formula we derive formulas for the second and third coefficients in the power series of functions in $\mathrm{SCC}(\alpha)$ in terms of coefficients of functions in $\mathscr{P}$ and CV. Suppose

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$
$$\varphi(z) = z + b_2 z^2 + b_3 z^3 + \cdots,$$

and

$$p(z) = 1 + d_1 z + d_2 z^2 + \cdots .$$

Then

$$\begin{aligned}
&\left\{e^{i\theta}[\cos(\theta) p(z) - i\sin(\theta)]\right\}^{\alpha} \\
&\quad = \left\{1 + e^{i\theta}\cos(\theta)[p(z) - 1]\right\}^{\alpha} \\
&\quad = 1 + \alpha\cos(\theta) e^{i\theta} d_1 z \\
&\qquad + \left[\alpha\cos(\theta) e^{i\theta} d_2 + \frac{\alpha(\alpha-1)}{2}\cos^2(\theta) e^{2i\theta} d_1^2\right] z^2 + \cdots .
\end{aligned}$$

By using the representation formula we obtain

$$\begin{aligned}
&1 + 2a_2 z + 3a_3 z^2 + \cdots \\
&\quad = f'(z) \\
&\quad = \left(1 + 2b_2 z + 3b_3 z^3 + \cdots\right)\left(1 + \alpha\cos(\theta) e^{i\theta} d_1 z + \cdots\right) \\
&\quad = 1 + \left[2b_2 + \alpha\cos(\theta) e^{i\theta} d_1\right] z \\
&\qquad + \Bigg[3b_3 + \alpha\cos(\theta) e^{i\theta} d_2 + \frac{\alpha(\alpha-1)}{2}\cos^2(\theta) e^{2i\theta} d_1^2 \\
&\qquad\qquad + 2\alpha\cos(\theta) e^{i\theta} b_2 d_1\Bigg] z^2 + \cdots .
\end{aligned}$$

Consequently,

$$2a_2 = 2b_2 + \alpha\cos(\theta) e^{i\theta} d_1 \tag{1}$$

and

$$\begin{aligned}
3a_3 = {}& 3b_3 + 2\alpha\cos(\theta) e^{i\theta} b_2 d_1 + \alpha\cos(\theta) e^{i\theta} d_2 \\
& + \frac{\alpha(\alpha-1)}{2}\cos^2(\theta) e^{2i\theta} d_1^2.
\end{aligned} \tag{2}$$

Next, we make use of the fact that

$$q(z) = 1 + \frac{z\varphi''(z)}{\varphi'(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

belongs to $\mathscr{P}$ to express a_2 and a_3 in terms of the coefficients of two functions in $\mathscr{P}$. From

$$1 + \frac{z\varphi''(z)}{\varphi'(z)} = 1 + 2b_2 z + \left(6b_3 - 4b_2^2\right)z^2 + \cdots,$$

we obtain

$$c_1 = 2b_2,$$
$$c_2 = 6b_3 - 4b_2^2,$$

or

$$2b_2 = c_1,$$
$$6b_3 = c_1^2 + c_2.$$

If we substitute these expressions into (1) and (2) we obtain

$$2a_2 = c_1 + \alpha\cos(\theta)e^{i\theta}d_1, \tag{3}$$

$$3a_3 = \frac{1}{2}\left(c_2 + c_1^2\right) + \alpha\cos(\theta)e^{i\theta}c_1 d_1 + \alpha\cos(\theta)e^{i\theta}d_2 + \frac{\alpha(\alpha-1)}{2}\cos^2(\theta)e^{2i\theta}d_1^2. \tag{4}$$

For $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathrm{SCC}(\alpha)$ the two functionals

$$L(f) = \mathrm{Re}\left\{a_2^2 - a_3\right\} + \tfrac{1}{3}|a_2|^2$$

and

$$M(f) = \mathrm{Re}\left\{3a_3 - 2a_2^2\right\} + 2|a_2|^2$$

play important roles in the proof of Theorem 1. We use (1) and (2) to express $L(f)$ in terms of b_1, b_2, d_1, and d_2. Now

$$a_2^2 - a_3 = b_2^2 - b_3 + \frac{\alpha}{3}\cos(\theta)e^{i\theta}b_2 d_1 - \frac{\alpha}{3}\cos(\theta)e^{i\theta}d_2 + \frac{\alpha(\alpha+2)}{12}\cos^2(\theta)e^{2i\theta}d_1^2$$

and

$$\frac{1}{3}|a_2|^2 = \frac{1}{3}|b_2|^2 + \frac{\alpha^2}{12}\cos^2(\theta)|d_1|^2 + \frac{\alpha}{3}\cos(\theta)\operatorname{Re}\left\{e^{i\theta}\overline{b}_2 d_1\right\},$$

so that

$$L(f) = \operatorname{Re}\left\{b_2^2 - b_3\right\} + \frac{1}{3}|b_2|^2 + \frac{2\alpha}{3}\cos(\theta)\operatorname{Re}\{b_2\}\operatorname{Re}\left\{e^{i\theta}d_1\right\} + \frac{\alpha}{3}\cos(\theta)\operatorname{Re}\left\{\frac{\alpha+2}{4}\cos(\theta)\left(e^{i\theta}d_1\right)^2 - e^{i\theta}d_2\right\} + \frac{\alpha^2}{12}\cos^2(\theta)|d_1|^2. \tag{5}$$

Similarly, we use (3) and (4) to obtain an expression for $M(f)$ in terms of c_1, c_2, d_1, and d_2. From

$$3a_3 - 2a_2^2 = \frac{1}{2}c_2 + \alpha\cos(\theta)e^{i\theta}d_2 - \frac{\alpha}{2}\cos^2(\theta)e^{2i\theta}d_1^2 \tag{6}$$

and

$$2|a_2|^2 = \frac{1}{2}|c_1|^2 + \alpha\cos(\theta)\operatorname{Re}\left\{e^{i\theta}\overline{c}_1 d_1\right\} + \frac{\alpha^2}{2}\cos^2(\theta)|d_1|^2, \tag{7}$$

we obtain

$$M(f) = \frac{1}{2}\operatorname{Re}\{c_2\} + \frac{1}{2}|c_1|^2 + \alpha\cos(\theta)\operatorname{Re}\left\{e^{i\theta}\overline{c}_1 d_1\right\} + \alpha\cos(\theta)\operatorname{Re}\left\{-\frac{1}{2}\cos(\theta)e^{2i\theta}d_1^2 + e^{i\theta}d_2\right\} + \frac{\alpha^2}{2}\cos^2(\theta)|d_1|^2. \tag{8}$$

3. TECHNICAL RESULTS

In this section we gather together several preliminary estimates that we need for our proof of Theorem 1.

LEMMA 1. *Suppose* $p(z) = 1 + d_1 z + d_2 z^2 + \cdots \in \mathscr{P}$ *and* $\theta \in (-\pi/2, \pi/2)$. *Then*

$$\cos(\theta)\left[2\,\mathrm{Re}\left\{e^{i\theta}d_1\right\} + \mathrm{Re}\left\{\tfrac{1}{2}\cos(\theta)\left(e^{i\theta}d_1\right)^2 - e^{i\theta}d_2\right\}\right] \le 4.$$

Equality holds if and only if $\theta = 0$ *and* $p(z) = (1+z)/(1-z)$.

Proof. Since $p(z) \in \mathscr{P}$ if and only if

$$p(e^{-i\theta}z) = 1 + e^{-i\theta}d_1 z + e^{-2i\theta}d_2 z^2 + \cdots$$

belongs to $\mathscr{P}$, it suffices to show that

$$N(p) = \cos(\theta)\left[2\,\mathrm{Re}\{d_1\} + \mathrm{Re}\left\{\tfrac{1}{2}\cos(\theta)d_1^2 - e^{-i\theta}d_2\right\}\right] \le 4$$

and equality implies $\theta = 0$ and $p(z) = (1+z)/(1-z)$.

Next, we eliminate d_2 in $N(p)$. Since $p \in \mathscr{P}$ [4],

$$\left|d_2 - \tfrac{1}{2}d_1^2\right| \le 2 - \tfrac{1}{2}|d_1|^2.$$

This yields

$$-\mathrm{Re}\left\{e^{-i\theta}d_2\right\} \le 2 - \tfrac{1}{2}|d_1|^2 - \tfrac{1}{2}\,\mathrm{Re}\left\{e^{-i\theta}d_1^2\right\},$$

so that

$$\begin{aligned}\mathrm{Re}\left\{\tfrac{1}{2}\cos(\theta)d_1^2 - e^{-i\theta}d_2\right\} &\le 2 - \tfrac{1}{2}|d_1|^2 + \tfrac{1}{2}\,\mathrm{Re}\left\{\left(\cos(\theta) - e^{-i\theta}\right)d_1^2\right\}\\ &= 2 - \tfrac{1}{2}|d_1|^2 - \tfrac{1}{2}\sin(\theta)\,\mathrm{Im}\left\{d_1^2\right\}.\end{aligned}$$

Thus,

$$N(p) \le \cos(\theta)\left[2\,\mathrm{Re}\{d_1\} + 2 - \tfrac{1}{2}|d_1|^2 - \tfrac{1}{2}\sin(\theta)\mathrm{Im}\left\{d_1^2\right\}\right].$$

Recall that if $p \in \mathscr{P}$, then $|d_1| \le 2$. Thus, we wish to show that

$$\cos(\theta)\left[2\,\mathrm{Re}\{d_1\} + 2 - \tfrac{1}{2}|d_1|^2 - \tfrac{1}{2}\sin(\theta)\mathrm{Im}\left\{d_1^2\right\}\right] \le 4$$

for $|d_1| \le 2$. Set $d_1 = 2re^{it}$, where $0 \le r \le 1$ and $t \in (-\pi, \pi]$. Then we want to demonstrate that

$$\begin{aligned}H(r) &= \cos(\theta)\left[4r\cos(t) + 2 - r^2(2 + 2\sin(\theta)\sin(2t))\right]\\ &= 2\cos(\theta)\left[2r\cos(t) + 1 - r^2(1 + 2\sin(\theta)\cos(t)\sin(t))\right] \le 4\end{aligned}$$

for $0 \le r \le 1$. Note that

$$H(0) = 2\cos(\theta) \le 2 < 4.$$

Next, we show that $H(1) \le 4$ with equality only for $\theta = t = 0$.

$$H(1) = 4\cos(\theta)\cos(t) - 4\cos(\theta)\sin(\theta)\cos(t)\sin(t).$$

We want to prove that

$$-\cos(\theta)\sin(\theta)\cos(t)\sin(t) \le 1 - \cos(\theta)\cos(t)$$

and that equality implies $\theta = t = 0$. Set $x = \cos(\theta) \in (0,1]$ and $y = \cos(t) \in [-1,1]$. Then $\pm\sqrt{1-x^2} = \sin(\theta)$ and $\pm\sqrt{1-y^2} = \sin(t)$. We want to show

$$\pm xy\sqrt{1-x^2}\sqrt{1-y^2} \le 1 - xy.$$

For $-1 \le y < 0$ and $0 < x \le 1$ it is elementary to see that strict inequality holds, so we need only consider $0 \le y \le 1$ and $0 \le x \le 1$ and show

$$xy\sqrt{1-x^2}\sqrt{1-y^2} \le 1 - xy$$

with strict inequality unless $x = y = 1$. This is equivalent to

$$G(x,y) = 1 - 2xy + x^4y^2 + x^2y^4 - x^4y^4$$

being nonnegative on the unit square $0 \le x,\ y \le 1$ and $G(x,y) = 0$ only for $x = y = 1$. Now,

$$\begin{aligned}
G(0,y) &= 1 > 0 \\
G(1,y) &= 1 - 2y + y^2 = (1-y)^2 \ge 0 \\
G(x,0) &= 1 > 0 \\
G(x,1) &= 1 - 2x + x^2 = (1-x)^2 \ge 0,
\end{aligned}$$

so $G(x,y) \ge 0$ on the boundary of the square with equality only at $(1,1)$. It is not difficult to show that if

$$\frac{\partial G}{\partial x}(x,y) = 0 = \frac{\partial G}{\partial y}(x,y)$$

at an interior point of the square, then $y = x$. Since

$$G(x,x) = (1-x^2)^2(1-x^4) > 0$$

for $0 < x < 1$, we conclude that $G(x, y) \geq 0$ for $0 \leq x, y \leq 1$ and equality implies $x = y = 1$. This proves that $H(1) \leq 4$ with equality only for $\theta = t = 0$.

All that remains is to show $H(r) < 4$ for $0 < r < 1$. If $H'(r) \neq 0$ for $0 < r < 1$, we are done. Now,

$$H'(r) = 4\cos(\theta)\left[\cos(t) - r(1 + 2\sin(\theta)\cos(t)\sin(t))\right].$$

If $H'(t_0) = 0$, then

$$r_0 = \frac{\cos(t)}{1 + 2\sin(\theta)\cos(t)\sin(t)}.$$

Suppose $r_0 \in (0, 1)$. Then

$$H(r_0) = 2\cos(\theta)\left[1 + r_0\cos(\theta)\right] < 4$$

since $r_0 < 1$. This proves that $H(r) \leq 4$ and equality implies $r = 1$ and $\theta = t = 0$. It follows that $N(p) \leq 4$ and equality implies $\theta = 0$ and $p(z) = (1 + z)/(1 - z)$.

Next, we prove Theorem 1(ii) for a special subclass of $\mathrm{SCC}(\alpha)$.

LEMMA 2. *For $\theta \in (-\pi/2, \pi/2)$ and $s, t \in (-\pi, \pi]$ set*

$$\varphi_s(z) = \frac{z}{1 - e^{is}z} \in \mathrm{CV},$$

$$q_s(z) = 1 + \frac{z\varphi_s''(z)}{\varphi_s'(z)} = \frac{1 + e^{is}z}{1 - e^{is}z}$$

$$= 1 + 2e^{is}z + 2e^{2is}z^2 + \cdots \in \mathscr{P}$$

and

$$p_t(z) = \frac{1 + e^{i(t-\theta)}z}{1 - e^{i(t-\theta)}z} = 1 + 2e^{i(t-\theta)}z + 2e^{2i(t-\theta)}z^2 + \cdots \in \mathscr{P}.$$

If $f_{s,t}(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathrm{SCC}(\alpha)$ is determined from

$$f_{s,t}'(z) = \varphi_s'(z)\left\{e^{i\theta}\left[\cos(\theta)p_t(z) - i\sin(\theta)\right]\right\}^{\alpha},$$

then

$$\left|3a_3 - 2a_2^2\right| + 2|a_2|^2 \leq 3 + 4\alpha + 2\alpha^2$$

and equality holds if and only if $\theta = 0$ and $s = t$; that is, if and only if $f_{s,t}$ is a rotation of F_α, $f_{s,s}(z) = e^{-is}F_\alpha(e^{is}z)$

Proof. For $f_{s,t}$ we have

$$c_1 = 2e^{is} \qquad d_1 = 2e^{i(t-\theta)}$$
$$c_2 = 2e^{2is} \qquad d_2 = 2e^{2i(t-\theta)}.$$

Then formulas (6) and (7) give

$$\begin{aligned}3a_3 - 2a_2^2 &= e^{2is} + 2\alpha\cos(\theta)e^{i(2t-\theta)} - 2\alpha\cos^2(\theta)e^{2it}\\ &= e^{2is}\left[1 - 2i\alpha\cos(\theta)\sin(\theta)e^{2iu}\right]\end{aligned}$$

and

$$\begin{aligned}2|a_2|^2 &= 2 + 4\alpha\cos(\theta)\operatorname{Re}\{e^{i(t-s)}\} + 2\alpha^2\cos^2(\theta)\\ &= 2\left[1 + 2\alpha\cos(\theta)\cos(u) + \alpha^2\cos^2(\theta)\right],\end{aligned}$$

where $u = t - s$. Note that

$$\left|3a_3 - 2a_2^2\right|^2 = 1 + 4\alpha\cos(\theta)\sin(\theta)\sin(2u) + 4\alpha^2\cos^2(\theta)\sin^2(\theta). \quad (9)$$

We wish to prove

$$\left|3a_3 - 2a_2^2\right| \le 3 + 4\alpha + 2\alpha^2 - 2|a_2|^2.$$

Since $f_{s,t} \in \mathrm{SCC}(\alpha)$, $|a_2| \le 1 + \alpha$ and so the right-hand side of the preceding inequality is positive. Therefore, it is enough to show

$$\begin{aligned}\left|3a_3 - 2a_2^2\right|^2 \le \left(3 + 4\alpha + 2\alpha^2 - 2|a_2|^2\right)^2 &= 1 + 8\alpha[1 - \cos(\theta)\cos(u)]\\ &\quad + 4\alpha^2\left[\sin^2(\theta) + 4(1 - \cos(\theta)\cos(u))^2\right]\\ &\quad + 16\alpha^3\sin^2(\theta)[1 - \cos(\theta)\cos(u)]\\ &\quad + 4\alpha^4\sin^4(\theta).\end{aligned}$$

From (9) we see that it is sufficient to show

$$\begin{aligned}&\cos(\theta)\sin(\theta)\sin(2u) + \alpha\cos^2(\theta)\sin^2(\theta)\\ &\quad\le 2[1 - \cos(\theta)\cos(u)] + \alpha\left[\sin^2(\theta) + 4(1 - \cos(\theta)\cos(u))^2\right]\\ &\qquad + 4\alpha^2\sin^2(\theta)[1 - \cos(\theta)\cos(u)] + \alpha^3\sin^4(\theta)\end{aligned}$$

and that equality implies $\theta = 0$ and $u = 0$ (that is, $s = t$). It is elementary that

$$\alpha \cos^2(\theta)\sin^2(\theta) \leq \alpha\left[\sin^2(\theta) + 4(1 - \cos(\theta)\cos(u))^2\right] + 4\alpha^2\sin^2(\theta)[1 - \cos(\theta)\cos(u)] + \alpha^3\sin^4(\theta)$$

and equality implies $\theta = 0$ and $u = 0$. Hence, all that remains is to verify that

$$\cos(\theta)\sin(\theta)\sin(2u) \leq 2[1 - \cos(\theta)\cos(u)]$$

or

$$\cos(\theta)\sin(\theta)\cos(u)\sin(u) \leq 1 - \cos(\theta)\cos(u).$$

If we replace θ by $-\theta$, then we see that this inequality was established in the proof of Lemma 1 and equality implies $\theta = 0 = u$. This completes the proof of Lemma 2.

COROLLARY. *For* $\theta \in (-\pi/2, \pi/2)$, $s, t \in (-\pi, \pi]$, *and* $0 < \alpha \leq 1$,

$$\cos(2s) + 4\alpha\cos(\theta)\cos(t - s) + 2\alpha\cos(\theta)\sin(\theta)\sin(2t) + 2\alpha^2\cos^2(\theta) \leq 1 + 4\alpha + 2\alpha^2.$$

Equality holds if and only if $\theta = 0$ *and* $s = t = 0, \pi$.

Proof. Since Theorem 1(ii) holds for $f_{s,t}$, we conclude that

$$M(f_{s,t}) \leq \left|3a_3 - 2a_2^2\right| + 2|a_2|^2 \leq 3 + 4\alpha + 2\alpha^2.$$

From formula (8) we find that

$$\begin{aligned} M(f_{s,t}) &= 2 + \cos(2s) + 4\alpha\cos(\theta)\cos(t - s) \\ &\quad + 2\alpha\cos(\theta)[\cos(2t - \theta) - \cos(\theta)\cos(2t)] \\ &\quad + 2\alpha^2\cos(\theta) \\ &= 2 + \cos(2s) + 4\alpha\cos(\theta)\cos(t - s) \\ &\quad + 2\alpha\cos(\theta)\sin(\theta)\sin(2t) + 2\alpha^2\cos^2(\theta). \end{aligned}$$

This proves the desired inequality. If equality holds, then equality must hold in Lemma 2 so $\theta = 0$ and $s = t$. Also,

$$\operatorname{Re}\left\{3a_3 - 2a_2^2\right\} = \left|3a_3 - 2a_2^2\right|.$$

For $\theta = 0$ and $s = t$ we have $3a_3 - 2a_2^2 = e^{2is}$, so the preceding equality implies $s = t = 0$ or $s = t = \pi$.

LEMMA 3. *Suppose* $\theta \in (-\pi/2, \pi/2)$, $s \in (-\pi, \pi]$, $\alpha > 0$, *and* $|d_1| \leq 2$. *Then*

$$\begin{aligned}\cos(2s) + 2\alpha(\cos(\theta)\mathrm{Re}\{e^{i(\theta-s)}d_1\} + 2\alpha\cos(\theta).\\ -\tfrac{1}{2}\alpha\cos(\theta)|d_1|^2 + \tfrac{1}{2}\alpha^2\cos^2(\theta)|d_1|^2\\ +\tfrac{1}{2}\alpha\cos(\theta)\sin(\theta)\mathrm{Im}\left\{\left(e^{i\theta}d_1\right)^2\right\} \leq 1 + 4\alpha + 2\alpha^2.\end{aligned}$$

Equality implies $\theta = 0$, *and either* $s = 0$ *and* $d_1 = 2$ *or* $s = \pi$ *and* $d_1 = -2$.

Proof. Since $|e^{i\theta}d_1| = |d_1|$, it suffices to show

$$\begin{aligned}\cos(2s) + 2\alpha\cos(\theta)\mathrm{Re}\{e^{-is}d_1\} + 2\alpha\cos(\theta)\\ -\tfrac{1}{2}\alpha\cos(\theta)|d_1|^2 + \tfrac{1}{2}\alpha^2\cos(\theta)|d_1|^2\\ +\tfrac{1}{2}\alpha\cos(\theta)\sin(\theta)\mathrm{Im}\{d_1^2\} \leq 1 + 4\alpha + 2\alpha^2.\end{aligned}$$

Set $d_1 = 2re^{it}$, where $0 \leq r \leq 1$ and $t \in (-\pi, \pi]$. Then we want to show

$$\begin{aligned}&\cos(2s) + 4\alpha r\cos(\theta)\cos(s-t) + 2\alpha\cos(\theta)\\ &\quad - 2\alpha r^2\cos(\theta) + 2\alpha^2r^2\cos^2(\theta) + 2\alpha r^2\cos(\theta)\sin(\theta)\sin(2t)\\ &\quad \leq 1 + 4\alpha + 2\alpha^2.\end{aligned}$$

This inequality will hold if we can show

$$\begin{aligned}H(r) &= \cos(2s) + 4\alpha r\cos(\theta)\cos(s-t) + 2\alpha\cos(\theta)\\ &\quad - 2\alpha r^2\cos(\theta) + 2\alpha^2\cos^2(\theta) + 2\alpha r^2\cos(\theta)\sin(\theta)\sin(2t)\\ &\leq 1 + 4\alpha + 2\alpha^2\end{aligned}$$

for $0 \leq r \leq 1$, with strict inequality unless $r = 1$ and either $s = t = 0$ or $s = t = \pi$.

Now

$$\begin{aligned}H(0) &= \cos(2s) + 2\alpha\cos(\theta) + 2\alpha^2\cos^2(\theta)\\ &\leq 1 + 2\alpha + 2\alpha^2 < 1 + 4\alpha + 2\alpha^2\end{aligned}$$

and

$$\begin{aligned}H(1) &= \cos(2s) + 4\alpha\cos(\theta)\cos(s-t) + 2\alpha^2\cos^2(\theta)\\ &\quad +2\alpha\cos(\theta)\sin(\theta)\sin(2t) \leq 1 + 4\alpha + 2\alpha^2\end{aligned}$$

by the Corollary to Lemma 2 with the proper conditions for equality. All that remains is to prove $H(r) < 1 + 4\alpha + 2\alpha^2$ for $0 < r < 1$. If $H'(r) \neq 0$ for $0 < r < 1$, we are done. Suppose $H'(r_0) = 0$ for some $r_0 \in (0, 1)$. Then

$$r_0 = \frac{\cos(s - t)}{1 - \sin(\theta)\sin(2t)}$$

and as $0 < r_0 < 1$

$$\begin{aligned} H(r_0) &= \cos(2s) + 2\alpha\cos(\theta) + 2\alpha^2\cos^2(\theta) \\ &\qquad + 2\alpha r_0\cos(\theta)\cos(s - t) \\ &\leq \cos(2s) + 2\alpha\cos(\theta) + 2\alpha^2\cos^2(\theta) + 2\alpha r_0\cos(\theta) \\ &\leq 1 + 2\alpha(1 + r_0) + 2\alpha^2 \\ &< 1 + 4\alpha + 2\alpha^2. \end{aligned}$$

This completes the proof.

4. PROOF OF THEOREM 1(i)

Because $\mathrm{SCC}(\alpha)$ is rotationally invariant, it suffices to show that $f \in \mathrm{SCC}(\alpha)$ implies

$$L(f) \leq \tfrac{1}{3}(1 + 4\alpha + 2\alpha^2) \tag{10}$$

and equality implies $f(z) = F_\alpha(z)$ or $f(z) = -F_\alpha(-z)$.

We begin by showing that we may assume $\mathrm{Re}\{e^{i\theta}d_1\} \geq 0$ in the expression (5) for $L(f)$. Suppose $\mathrm{Re}\{e^{i\theta}d_1\} < 0$. Note that $g(z) = -f(-z) \in \mathrm{SCC}(\alpha)$. Also,

$$g'(z) = f'(-z) = \varphi'(-z)\left[e^{i\theta}(\cos(\theta)p(-z) - i\sin(\theta))\right]^\alpha$$

where

$$-\varphi(-z) = z - b_1 z + b_2 z^2 + \cdots$$

and

$$p(-z) = 1 - d_1 z + d_2 z^2 + \cdots .$$

Now, $\mathrm{Re}\{e^{i\theta}(-d_1)\} > 0$ and $L(f) = L(g)$; we replace f by $g(z) = -f(-z)$ when $\mathrm{Re}\{e^{i\theta}d_1\} < 0$. Thus, we want to prove (10) under the restriction $\mathrm{Re}\{e^{i\theta}d_1\} \geq 0$ and show that equality implies $f = F_\alpha$.

Because $\varphi \in \mathrm{CV}$, we have [7]

$$\operatorname{Re}\left\{b_2^2 - b_3\right\} + \tfrac{1}{3}|b_2|^2 \leq \tfrac{1}{3} \tag{11}$$

and

$$\operatorname{Re}\{b_2\} \leq |b_2| \leq 1.$$

Note that $\operatorname{Re}\{b_2\} = 1$ if and only if $p(z) = z/(1-z)$. Since $\operatorname{Re}\{e^{i\theta}d_1\} \geq 0$, we obtain

$$\begin{aligned} L(f) \leq \frac{1}{3} &+ \frac{2\alpha}{3}\cos(\theta)\operatorname{Re}\left\{e^{i\theta}d_1\right\} \\ &+ \frac{\alpha}{3}\cos(\theta)\operatorname{Re}\left\{\frac{1}{2}\cos(\theta)\left(e^{i\theta}d_1\right)^2 - e^{i\theta}d_2\right\} \\ &+ \frac{\alpha^2}{12}\cos^2(\theta)\left[|d_1|^2 + \operatorname{Re}\left\{\left(e^{i\theta}d_1\right)^2\right\}\right]. \end{aligned}$$

By Lemma 1,

$$\cos(\theta)\left[2\operatorname{Re}\left\{e^{i\theta}d_1\right\} + \operatorname{Re}\left\{\tfrac{1}{2}\cos(\theta)\left(e^{i\theta}d_1\right)^2 - e^{i\theta}d_2\right\}\right] \leq 4$$

and equality implies $\theta = 0$ and $p(z) = (1+z)/(1-z)$. Since $p \in \mathscr{P}$, $|d_1| \leq 2$. Thus,

$$L(f) \leq \frac{1}{3} + \frac{4\alpha}{3} + \frac{2}{3}\alpha^2\cos^2(\theta) \leq \frac{1}{3}(1 + 4\alpha + 2\alpha^2).$$

Equality implies $\theta = 0$, $p(z) = (1+z)/(1-z)$ and $\varphi(z) = z/(1-z)$, that is, $f = F_\alpha$. Conversely, if $f = F_\alpha$ it is straightforward to verify that equality holds.

5. PROOF OF THEOREM 1(ii)

Since $\mathrm{SCC}(\alpha)$ is rotationally invariant, it is enough to prove that $f \in \mathrm{SCC}(\alpha)$ implies

$$M(f) \leq 3 + 4\alpha + 2\alpha^2$$

with equality if and only if either $f(z) = F_\alpha(z)$ or $f(z) = -F_\alpha(-z)$.

We begin by bounding $M(f)$ by a quantity depending only on c_1 and d_1. Since $q \in \mathscr{P}$ [4],

$$\left|d_2 - \tfrac{1}{2}d_1^2\right| \leq 2 - \tfrac{1}{2}|d_1|^2$$

and so

$$\operatorname{Re}\left\{e^{i\theta}d_2\right\} \leq 2 - \tfrac{1}{2}|d_1|^2 + \tfrac{1}{2}\operatorname{Re}\left\{e^{i\theta}d_1^2\right\}.$$

Then

$$\operatorname{Re}\left\{-\tfrac{1}{2}\cos(\theta)\left(e^{i\theta}d_1\right)^2 + e^{i\theta}d_2\right\} \le 2 - \tfrac{1}{2}|d_1|^2 + \tfrac{1}{2}\sin(\theta)\operatorname{Im}\left\{\left(e^{i\theta}d_1\right)^2\right\}.$$

This yields

$$\begin{aligned} M(f) \le{}& \frac{1}{2}\operatorname{Re}\{c_2\} + \frac{1}{2}|c_1|^2 + \alpha\cos(\theta)\operatorname{Re}\left\{e^{i\theta}\bar{c}_1 d_1\right\} \\ &+ 2\alpha\cos(\theta) - \frac{\alpha}{2}\cos(\theta)|d_1|^2 \\ &+ \frac{\alpha}{2}\cos(\theta)\sin(\theta)\operatorname{Im}\left\{\left(e^{i\theta}d_1\right)^2\right\} + \frac{\alpha^2}{2}\cos^2(\theta)|d_1|^2. \end{aligned}$$

Similarly, $p \in \mathscr{P}$ so

$$\left|c_2 - \tfrac{1}{2}c_1^2\right| \le 2 - \tfrac{1}{2}|c_1|^2,$$

or

$$\begin{aligned} \operatorname{Re}\{c_2\} &\le 2 - \tfrac{1}{2}|c_2|^2 + \tfrac{1}{2}\operatorname{Re}\{c_1^2\}, \\ \tfrac{1}{2}\operatorname{Re}\{c_2\} + \tfrac{1}{2}|c_1|^2 &\le 1 + \tfrac{1}{4}|c_1|^2 + \tfrac{1}{4}\operatorname{Re}\{c_1^2\} \\ &= 1 + \tfrac{1}{2}\operatorname{Re}^2\{c_1\}. \end{aligned}$$

Thus,

$$\begin{aligned} M(f) \le{}& 1 + \frac{1}{2}\operatorname{Re}^2\{c_1\} + \alpha\cos(\theta)\operatorname{Re}\left\{e^{i\theta}\bar{c}_1 d_1\right\} \\ &+ 2\alpha\cos(\theta) - \frac{\alpha}{2}\cos(\theta)|d_1|^2 \\ &+ \frac{\alpha}{2}\cos(\theta)\sin(\theta)\operatorname{Im}\left\{\left(e^{i\theta}d_1\right)^2\right\} + \frac{\alpha^2}{2}\cos^2(\theta)|d_1|^2 = \tilde{M}(f). \end{aligned}$$

Next, we show $\tilde{M}(f) \le 3 + 4\alpha + 2\alpha^2$ when $c_1 = 2e^{is}$. For $c_1 = 2e^{is}$,

$$\begin{aligned} \tilde{M}(f) ={}& 1 + 2\cos^2(s) + 2\alpha\cos(\theta)\operatorname{Re}\left\{e^{i(\theta-s)}d_1\right\} \\ &+ 2\alpha\cos(\theta) - \frac{\alpha}{2}\cos(\theta)|d_1|^2 \\ &+ \frac{\alpha}{2}\cos(\theta)\sin(\theta)\operatorname{Im}\left\{\left(e^{i\theta}d_1\right)^2\right\} + \frac{\alpha^2}{2}\cos^2(\theta)|d_1|^2. \end{aligned}$$

Since

$$1 + 2\cos^2(s) = 2 + \cos(2s),$$

the inequality $\tilde{M}(f) \leq 3 + 4\alpha + 2\alpha^2$ follows immediately from Lemma 3. Equality implies $\theta = 0$ and either $s = 0$ $(c_1 = 2)$ and $d_1 = 2$ or $s = \pi(c_1 = -2)$ and $d_1 = -2$. This means either $f(z) = F_\alpha(z)$ or $f(z) = -F_\alpha(-z)$.

All that remains is to demonstrate $\tilde{M}(f) < 3 + 4\alpha + 2\alpha^2$ for $|c_1| < 2$. Since $\tilde{M}(f)$ is a subharmonic function of c_1 which is nonconstant and $\tilde{M}(f) \leq 3 + 4\alpha + 2\alpha^2$ for $|c_1| = 2$, we get strict inequality for $|c_1| < 2$.

6. INVARIANT FORMULATION

A not necessarily normalized analytic function f on $\mathbb{D}$ is called strongly close-to-convex of order α if there exists a not necessarily normalized convex univalent function φ on $\mathbb{D}$ such that

$$\left| \arg \frac{f'(z)}{\varphi'(z)} \right| < \frac{\alpha\pi}{2}$$

for $z \in \mathbb{D}$. This class of functions is linearly invariant in the sense that

$$\frac{f(T(z)) - f(T(0))}{f'(T(0))T'(0)} \in \mathrm{SCC}(\alpha)$$

for any conformal automorphism T of $\mathbb{D}$. Pommerenke [6] noted this linear invariance and gave a simple external geometric characterization when $0 < \alpha \leq 1$.

There is an invariant formulation of Theorem 1 for nonnormalized strongly close-to-convex functions of order α. Recall the differential operators [3]

$$D_1 f(z) = (1 - |z|^2) f'(z)$$

$$D_2 f(z) = (1 - |z|^2)^2 f''(z) - 2\bar{z}(1 - |z|^2) f'(z)$$

$$D_3 f(z) = (1 - |z|^2)^3 f'''(z) - 6\bar{z}(1 - |z|^2)^2 f''(z)$$
$$+ 6\bar{z}^2(1 - |z|^2) f'(z).$$

Note that $D_j f(0) = f^{(j)}(0)$ $(j = 1, 2, 3)$. These differential operators are invariants in the sense that $|D_j(S \circ f \circ T)| = |D_j(f) \circ T|$ for all Euclidean motions S of $\mathbb{C}$ and conformal automorphisms T of $\mathbb{D}$. The Schwarzian derivative

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is related by

$$\frac{D_3 f(z)}{D_1 f(z)} - \frac{3}{2}\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2 = (1 - |z|^2)^2 S_f(z).$$

THEOREM 2. *Suppose f is strongly close-to-convex of order α on $\mathbb{D}$. Then*

$$(1 - |z|^2)|S_f(z)| + \frac{1}{2}\left|\frac{D_2 f(z)}{D_1 f(z)}\right|^2 \leq 2(1 + 4\alpha + 2\alpha^2)$$

and

$$\left|\frac{D_3 f(z)}{D_1 f(z)} - \left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2\right| + \left|\frac{D_2 f(z)}{D_1 f(z)}\right|^2 \leq 2(3 + 4\alpha + 2\alpha^2).$$

These inequalities are both sharp; equality holds if and only if $f = S \circ f_\alpha \circ T$ where S is a conformal automorphism of $\mathbb{C}$ and T is a conformal automorphism of $\mathbb{D}$.

Proof. For $z = 0$ these inequalities reduce to Theorem 1. The general case follows from the invariance of the differential operators together with the linear invariance of the class of functions.

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